

Fast State Transfer and Entanglement Renormalization Using Long-Range Interactions

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In short-range interacting systems, the speed at which entanglement can be established between two separated points is limited by a constant Lieb-Robinson velocity. Long-range interacting systems are capable of faster entanglement generation, but the degree of the speed-up possible is an open question. In this paper, we present a protocol capable of transferring a quantum state across a distance L in d dimensions using long-range interactions with strength bounded by $1/r^\alpha$. If $\alpha < d$, the state transfer time is asymptotically independent of L ; if $\alpha = d$, the time is logarithmic in distance L ; if $d < \alpha < d+1$, transfer occurs in time proportional to $L^{\alpha-d}$; and if $\alpha \geq d+1$, it occurs in time proportional to L . We then use this protocol to upper bound the time required to create a state specified by a MERA (multiscale entanglement renormalization ansatz) tensor network, and show that, if the linear size of the MERA state is L , then it can be created in time that scales with L identically to state transfer up to multiplicative logarithmic corrections.

Entanglement generation in a quantum system is limited, even in a non-relativistic setting, by the interactions available. In a system with short-range interactions, Lieb and Robinson showed that there exists a linear light cone defined by a speed proportional to both the interaction range and strength [1]. If a system initially in a product state begins evolving under an interacting Hamiltonian, correlations decrease exponentially outside of this causal cone [2–4]. However, in physical systems such as polar molecules, Rydberg atoms, or trapped ions, the interactions are not of limited range, but rather fall off with distance r as a power law $1/r^\alpha$. For these systems, generalizations of the Lieb-Robinson bound are known, but they may not be tight [5–7]. In addition, for sufficiently long-ranged interactions the cone may even encompass infinite space at finite time, signaling a breakdown of emergent locality [8–11].

These bounds on entanglement have direct applications to quantum information processing. The Lieb-Robinson bound limits the speed at which the operations can be performed or states created [12]. Creating highly entangled states has important applications in metrology and communication [13–17]. In this paper, we consider the task of using long-range interactions to speed up certain quantum information processes such as quantum state transfer, GHZ (Greenberger-Horne-Zeilinger) state preparation, and MERA (multiscale entanglement renormalization ansatz) construction.

State transfer is a process by which an unknown quantum state on one site in a lattice is transferred to another site [18–21]. Discussion of possible experimental realizations can be found in Refs. [22–24]. Since state transfer establishes perfect correlation between a site at $t = 0$ and another site after the transfer, it is limited by the Lieb-Robinson bound in the system under consideration. In this work, we propose a state transfer protocol which

makes use of long-range interactions to transfer a state a distance L on a d -dimensional lattice in time proportional to L^0 ($\alpha < d$), $\log L$ ($\alpha = d$), $L^{\alpha-d}$ ($d < \alpha \leq d+1$), or L ($\alpha \geq d$). As an intermediate step of the protocol presented, a many-body state resembling the GHZ state is created; the formation speed of such a highly correlated state is also limited by the Lieb-Robinson bound [12]. Rather than act in a pairwise fashion on qubits, we use many qubits at once to complete quantum operations more quickly.

As we will discuss, one powerful application of fast state transfer using long-ranged interactions would be a fast realization of a circuit described by a MERA [25–27]. MERAs are particularly useful ways to represent entangled states [28–30], such as the ground states of the toric code, topological insulators, and quantum Hall states [31–33]. MERAs may also offer insight into fundamental concepts such as holography [34]. By performing state transfer and then applying a two-qubit gate between nearest neighbors, we can speed up long-range two-qubit gates, which we use to upper bound the minimal time required to create a state from a MERA.

State Transfer.—Our state transfer protocol first creates a many-body entangled state including the intended starting and final qubits. We do so by applying a controlled X rotation between pairs of qubits (i, j) using a Hamiltonian

$$H_{i,j} = h_{ij} (|0\rangle\langle 0|_i \otimes I_j + |1\rangle\langle 1|_i \otimes X_j). \quad (1)$$

Here h_{ij} is the interaction strength, which may not be identical for all pairs of qubits. I_j and X_j are the identity and Pauli X operator acting on qubit j . Once Eq. (1) is applied for a time $t = \pi/(2h_{ij})$, it effects a controlled-NOT (CNOT) gate between qubits i and j (up to an unimportant phase). In Eq. (1), i is the control qubit for the CNOT while j is the target qubit. When applied to

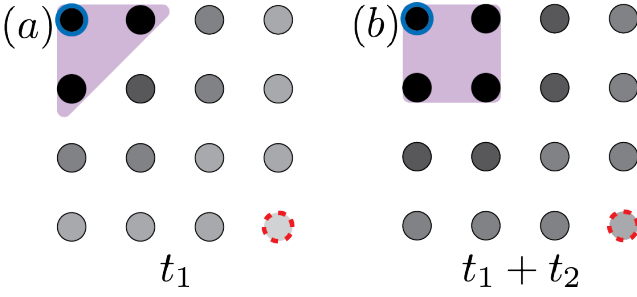


FIG. 1. Our state transfer protocol using long-range interactions. We want to move a qubit state from the upper-left site (outlined in solid blue) to the lower-right one (outlined in dashed red) After a time t_1 (a), the nearest-neighbor qubits have shifted from target to control (purple region), and continue acting on all other qubits, thereby adding an additional qubit to the set of controls after further time t_2 (b). Each qubit rotates further (shown by darker shading). The growth continues until the original qubit has effectively performed a CNOT on all qubits in the lattice shown.

a control qubit in an arbitrary state and a target qubit in the state $|0\rangle$, the CNOT gate results in a two-qubit state encoding the original qubit,

$$\text{CNOT}(\alpha|0\rangle + \beta|1\rangle)|0\rangle = \alpha|00\rangle + \beta|11\rangle. \quad (2)$$

By continuing this process, we can create a many-body entangled state of N qubits $\alpha|0\rangle^{\otimes N} + \beta|1\rangle^{\otimes N}$ encoding the same state as the initial qubit. The original state can be transferred onto target qubit by reversing the entangling process and leaving the destination qubit as the final control qubit. If the entangled state is constructed to be symmetric in space between the original qubit and the destination qubit in state transfer, then it follows that the transfer time is twice the time required to construct the state. Suppose that $H_{i,j}$ were a nearest-neighbor Hamiltonian; this procedure would then allow us to transfer a qubit state a distance L by applying L CNOT operations to construct the many-body state and then L other CNOT operations which are properly time-reversed and spatially mirrored. The requirement of a time at least linear in L to perform state transfer in this case is a direct consequence of the Lieb-Robinson bound.

However, by using Hamiltonians with long-range interactions, we can achieve a faster state transfer. We suppose that $h_{ij} = 1/r_{ij}^\alpha$, where r_{ij} is the distance between the qubits i and j [35]. Our protocol (Fig. 1) starts by acting on all qubits in the lattice with a single control qubit storing the initial state. Once the CNOT operation completes on a qubit, it can be switched from a target to a control, and then used to speed up the CNOTs which are still continuing on other qubits. If a single qubit is targeted by many control qubits, then the CNOT operation on that qubit can be completed faster. (Multiple $H_{i,j}$ will mutually commute as long as the sets of target qubits and control qubits are disjoint.) If qubit j is tar-

geted by many qubits indexed by i , the time required to complete the CNOT becomes

$$t = \frac{\pi}{2 \sum_i h_{ij}} = \frac{\pi}{2 \sum_i r_{ij}^{-\alpha}}. \quad (3)$$

In addition to the progressive inclusion of more control qubits, each subsequent qubit has already been rotated by some angle, reducing the remaining time required to complete the operation. Therefore, additional qubits can be added more quickly to the state as it grows.

As an example, consider beginning with a system of three qubits arranged in a line,

$$|\psi(t=0)\rangle = (\alpha|0\rangle + \beta|1\rangle)|00\rangle. \quad (4)$$

Simultaneously applying $H_{1,2}$ and $H_{1,3}$ for a time $t_1 = \pi/2$, the state becomes

$$|\psi(t_1)\rangle = \alpha|000\rangle - i\beta|11\rangle \left(\cos \frac{\pi}{2\alpha+1} |0\rangle - i \sin \frac{\pi}{2\alpha+1} |1\rangle \right). \quad (5)$$

At this point, the second qubit is made a control, so the acting Hamiltonians are $H_{1,3}$ and $H_{2,3}$. By continuing the evolution under these Hamiltonians for an additional time,

$$t_2 = \frac{\frac{\pi}{2} - \frac{\pi}{2\alpha+1}}{1 + \frac{1}{2\alpha}} = \frac{\text{rotation remaining}}{\text{sum of interactions}}, \quad (6)$$

the system will end in the final state

$$|\psi(t_1 + t_2)\rangle = \alpha|000\rangle - \beta|111\rangle. \quad (7)$$

The entire procedure can be reversed, interchanging the roles of qubits 1 and 3, to transfer the original state,

$$|\psi(2(t_1 + t_2))\rangle = |00\rangle(\alpha|0\rangle + \beta|1\rangle). \quad (8)$$

We now consider the case of many qubits. The total time required to continue growing the state to a given distance will depend on both α and the dimensionality of the system, d . Our strategy will be to specify a suboptimal protocol that simplifies the calculation of the time required for state transfer, and then use that result to bound (from above) the state transfer time of the protocol in Fig. 1, which is more difficult to analyze directly.

First, we specify that we aim to construct a GHZ state across a hypercube whose diagonal spans a distance $L\sqrt{d}$ (see Fig. 2), and the points on either end of the diagonal are the original and destination sites for state transfer. Because the state transfer time using the protocol of Fig. 1 is difficult to compute, we use a slightly slower protocol that allows us to easily estimate the transfer time both analytically and numerically. Rather than change a qubit into a control as soon as its evolution completes, we instead halt a qubit's evolution when its rotation finishes. Once we have enough qubits to form a full hypercube of controls, we expand the control set and continue evolution. This scheme is illustrated in Fig. 2, and we expect

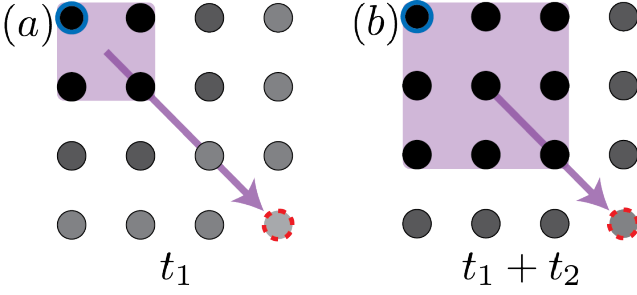


FIG. 2. (a) The suboptimal protocol used for our bounds, with the same color scheme as Fig. 1. After the j th time step, a $(j+1) \times (j+1)$ hypercube of qubits act as controls. The purple arrow represents $H(2,3)$, as it connects a 2×2 square to a qubit at coordinates $(3,3)$. (b) After time $t_1 + t_2$ another set of qubits has been converted from targets to controls. The purple arrow now represents $H(3,3)$.

it to perform similarly (in terms of the scaling of transfer time) with the scheme in Fig. 1. Let $k = 1, 2 \dots L$ denote each subsequent expansion of the hypercube, so that after time $t = t_1 + t_2 \dots + t_k$ we can form a complete control hypercube of edge length k . The times t_k are determined by the condition that each qubit must accumulate a total phase of $\pi/2$,

$$\sum_{i=1}^k H(i, k) t_i = \frac{\pi}{2}. \quad (9)$$

Here $H(r, s)$ is defined to be the summation of all Hamiltonians for which i is in the hypercube with corners $(0, 0, 0, \dots)$ and (r, r, r, \dots) and j is at the site (s, s, s, \dots) at the corner of a larger hypercube containing the first, as illustrated in Fig. 2.

At this point, we will begin looking for bounds on the times t_k . Our first bound arises by noting that for all k , $t_k > t_{k+1}$. This is because, for each k , the quantity $H(k, k)$ is strictly larger – the qubit at (k, k, \dots, k) has more qubits acting on it than its counterpart in the previous step. We use $t_k > t_{k+1}$ to rewrite the phase condition on times in Eq. (9),

$$\frac{\pi}{2} \geq t_k \sum_{i=1}^k H(i, k). \quad (10)$$

We now construct two complementary bounds for $H(i, k)$. In some cases (small α), $H(i, k)$ will receive appreciable contributions from the entire hypercube of control qubits. In this case, we can obtain a lower bound by pretending that all control qubits are at the same point a distance $k\sqrt{d}$ away, the maximum possible. However, for large α , the interaction is dominated by nearby qubits whose contribution is independent of k . For instance, in $H(k, k)$ there is always one qubit at the nearest vertex of the hypercube whose contribution does not depend on k .

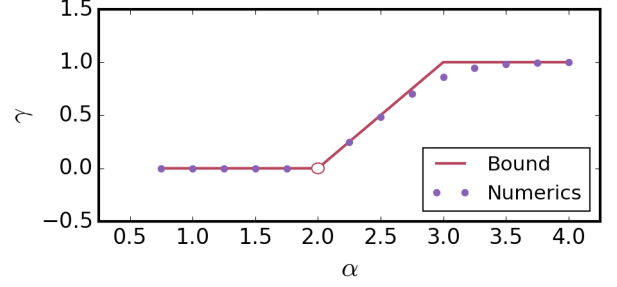


FIG. 3. Numerical results of solving Eq. (9) at different α in $d = 2$. Points show the β resulting from a fit of $\sum_{i \leq L} t_i$ to L^β for L between 900 and 1000. The solid line shows the bound from Eq. (12). At $\alpha = d$ (open circle), the numerics are consistent with the expected logarithmic scaling; the fact that the bound is not saturated at $\alpha = 3$ is due to finite L and should vanish in the $L \rightarrow \infty$ limit.

These two bounds can be combined to yield:

$$H(i, k) \geq \max \left(\frac{i^d}{(k\sqrt{d})^\alpha}, \frac{\delta_{ik}}{d^{\alpha/2}} \right). \quad (11)$$

After substituting Eq. (11) into Eq. (10), the sum can be performed. If we discard all constants depending only on d or α , the result is a bound on t_k ,

$$t_k \lesssim \min \left(k^{\alpha-(d+1)}, 1 \right). \quad (12)$$

To obtain the scaling of the entire state transfer process (which is twice the time required to create the GHZ state), a sum over t_k is made up to $k = L$. For $\alpha < d$, $t_k \lesssim k^{-1}$, so the sum converges to a constant for asymptotic k . The convergence signals that a state can be transferred any desired distance in a constant time. For $\alpha = d$, $t_k = k^{-1}$, so the sum scales logarithmically in L . For $d < \alpha < d+1$, we obtain a polynomial scaling $L^{\alpha-d}$. Finally, for $\alpha \geq d+1$, the constant lower bound on t_k dominates, and state transfer takes a time proportional to L , just as it does for short-range interacting systems. These scalings are illustrated in Fig. 3, along with polynomial fits of the numerical solutions of Eq. (9).

Constructing a MERA.—We now demonstrate that our state transfer protocol allows for construction of a MERA in a time that, up to multiplicative logarithmic corrections, scales with L , the linear system size, as state transfer scales for a distance L . A MERA can be viewed as a series of transformations between increasingly coarse-grained lattices, removing local entanglement at each step.

Suppose we begin with a lattice \mathcal{L}_0 with lattice constant 1, where the state of each site is specified in a vector space of dimension χ (see Fig. 4). First, we group sites in \mathcal{L}_0 into supersites, each of which consists of n sites. A MERA consists of two alternating types of unitary operations. The first type of unitary, called a disentangler,

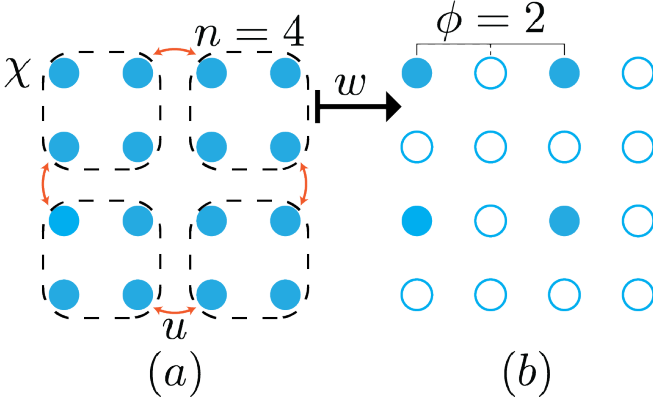


FIG. 4. MERA operations and parameters. (a) Disentanglers u act between neighboring supersites of $n = 4$ individual sites to remove local entanglement, then we move to (b) by applying an isometry w on each supersite. After these operations, the length scale in the problem has been doubled. Filled-in qubits remain in the lattice after this layer of the MERA. Others are disentangled and left in the state $|0\rangle$.

acts between neighboring supersites. These operations remove entanglement at the length scale of \mathcal{L}_0 . The next operation, an isometry, maps each supersite into a single site with dimension χ while leaving other sites in the state $|0\rangle$ [36]. The result is a new lattice, \mathcal{L}_1 , which is a coarse-graining of \mathcal{L}_0 . The coarse-graining can continue, defining a sequence of lattices $\mathcal{L}_0, \mathcal{L}_1, \dots, \mathcal{L}_S$. The final lattice \mathcal{L}_S possesses just a few sites whose joint state is specified by the top tensor. The number of layers S will depend on the linear system size L of the original lattice, and the factor $\phi = n^{1/d}$ by which the length scale changes at each MERA layer: $S = \log_\phi L$. Note that while we have presented a MERA as renormalizing a many-body state, the reverse process allows us to build the state in \mathcal{L}_0 from a product state.

Disentanglers act between two supersites of dimensions χ^n , so they are unitary operations on a χ^{2n} -dimensional vector space, while isometries only act on a single supersite, a χ^n -dimensional vector space. Since implementing an arbitrary $r \times r$ unitary matrix requires $\sim r^2$ two-qubit gates [37], an arbitrary disentangler requires $\sim \chi^{4n}$ two-qubit gates while an arbitrary isometry requires only $\sim \chi^{2n}$ such gates.

Suppose that t_τ is the maximum time required to perform a two-qubit gate in the lattice \mathcal{L}_τ . Then we can say that the time required to perform disentangling operations will be $\sim \chi^{4n} t_\tau$, while the time required to perform the isometry is $\sim \chi^{2n} t_\tau$. The time to perform the entire MERA circuit will then be bounded (up to a multiplicative constant) by

$$t_{\text{MERA}} \lesssim (\chi^{4n} + \chi^{2n}) \sum_{\tau=0}^{S-1} t_\tau. \quad (13)$$

We can perform a two-qubit gate at a distance by

first performing state transfer, implementing the required two-qubit operation on neighboring qubits, and then reversing the state transfer. This procedure allows for $t_\tau = 2t_{\text{transfer}}$. The time required to perform the final two-qubit gate does not affect the scaling. We can then bound t_{transfer} by considering the largest length scale in each lattice, $\ell_\tau = \phi^\tau$. If $\alpha = d$, t_{transfer} scales as $\log \ell_\tau$ (as in the previous section, but with a constant multiple changing the base of the logarithm), and t_{MERA} will be bounded by $\sim (\log L)^2$ by considering the largest term in Eq. (13) multiplied by the number of terms. For $\alpha \neq d$, t_{transfer} scales polynomially in ℓ_τ with exponent β ,

$$t_{\text{MERA}} \lesssim (\chi^{4n} + \chi^{2n}) \sum_{\tau=0}^{S-1} \ell_\tau^\beta. \quad (14)$$

For $\alpha < d$, $\beta = 0$ and the sum is proportional to $\log L$. For $\alpha \neq d$, we use $\ell_\tau = \phi^\tau$ and carry out the geometric sum to obtain

$$t_{\text{MERA}} \lesssim (\chi^{4n} + \chi^{2n}) (\phi^\beta)^S = (\chi^{4n} + \chi^{2n}) L^\beta. \quad (15)$$

Thus we have

$$t_{\text{MERA}} \lesssim \begin{cases} \log L & \alpha < d \\ \log^2 L & \alpha = d \\ L^{\alpha-d} & d < \alpha \leq d+1 \\ L & \alpha > d+1. \end{cases} \quad (16)$$

Outlook.—We have demonstrated fast state transfer and MERA construction protocols using long-range interactions. However, we have not shown that our method is the fastest state transfer protocol possible using interaction strengths bounded by $1/r^\alpha$. Such a result would require demonstrating a general Lieb-Robinson-type bound which we would then saturate. However, a state transfer protocol with the fast scalings we have presented here serves as an example that limits future Lieb-Robinson bounds. The state transfer protocol we have presented establishes that no finite causal region is possible for $\alpha < d$, since a constant amount of time suffices to establish any desired correlation at arbitrary distances. In previous work, causal regions were seen in systems with $d/2 < \alpha \leq d$ as long as the initial state was not entangled [8]; we have shown that this is not true in general. These results should be compared to Ref. [7], which established a polynomial light cone only for $\alpha > 2D$ that becomes linear only in the limit of $\alpha \rightarrow \infty$. In addition, the fact that our protocol gives the same state-transfer time scaling ($\sim L$) as a short-range system when $\alpha \geq d+1$ suggests that the tightest possible Lieb-Robinson bound may also possess a critical alpha with a similar property.

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